# Scalable methods for solving games in Adversarial Machine Learning

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# Adversarial Machine Learning

- Study and guarantee **robustness** of ML-based decisions wrt adversarial data manipulation.
- Conflict adversary - learning system modeled as a **game**.
- Classical Decision Makers, Humans: **discrete** and **low dimensional** decision spaces.
- New Decision Makers, Algorithms: **continuous** and **high dimensional** decision spaces.

**Scalable gradient-based methods for solving sequential games in the new paradigm**

# Motivation - Adversarial Regression

- $R_J$ and $R_D$ are two competing wine brands.
- $R_D$ has a system to automatically measure wine quality training a regression over some quality indicators. (Response value: wine quality, Covariates: quality indicators).
- $R_J$, aware of the actual superiority of its competitor's wines, decides to **hack** $R_D$'s system by manipulating the value of several quality indicators **at operation time**, to artificially decrease $R_D$'s quality rates.

# Motivation - Adversarial Regression

- $R_D$ is **aware** of the possibility of being hacked and decides to train its regression in an **adversarial robust** manner.
- $R_D$ models this **conflict** as a game between a *learner* $(R_D)$ and a *data generator* $(R_J)$ . (Brückner and Scheffer, 2011).
- The *data generator* tries to fool the learner **modifying input data at application time**, inducing a change between the data distribution at training $[p(x, y)]$ and test $[\bar{p}(x, y)]$ times.

# The Learner Problem

- Given a feature vector $\boldsymbol{x} \in \mathbb{R}^p$ and target $\boldsymbol{y} \in \mathbb{R}$, the learner's decision is to choose the weight vector of a linear model $f_w(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{w}$, minimizing **theoretical costs at application time**

$$\theta_l(w, \bar{p}, c_l) = \int c_l(x, y)(f_w(x) - y)^2 \, \mathrm{d}\bar{p}(x, y),$$

- To do so, the learner has a training matrix $\boldsymbol{X} \in \mathbb{R}^{n \times p}$ and a vector of target values $\boldsymbol{y} \in \mathbb{R}^n$ (a sample from distribution $\boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ at training time).

# The Data Generator Problem

- The data generator aims at **changing features of test instances** to induce a transformation from $p(x, y)$ to $\bar{p}(x, y)$.
- $z(x, y)$ is the data generator's target value for instance $x$ with real value $y$
- The data generator aims at **choosing the data transformation** that minimizes the theoretical costs given by

$$\theta_d(w, \bar{p}, c_d) = \int c_d(x, y)(f_w(x) - z(x, y))^2 \, \mathrm{d}\bar{p}(x, y) + \Omega_d(p, \bar{p})$$

# Regularized Empirical Costs

- Theoretical costs defined above depend on the unknown distributions $\boldsymbol{p}$ and $\bar{\boldsymbol{p}}$.
- We focus on their regularized empirical counterparts, given by

$$\hat{\theta}_l(w, \bar{X}, c_l) = \sum_{i=1}^{n} c_{l,i}(f_w(\bar{x}_i) - y_i)^2 + \Omega_l(f_w),$$
$$\hat{\theta}_d(w, \bar{X}, c_d) = \sum_{i=1}^{n} c_{d,i}(f_w(\bar{x}_i) - z_i)^2 + \Omega_d(X, \bar{X}).$$

# Resulting Stackelberg Game

- We assume the learner acts first, choosing a weight vector $\boldsymbol{w}$. Then the data generator, after observing $\boldsymbol{w}$, chooses his optimal data transformation.

$$
\begin{aligned}
\arg\min_{w} \quad & \widehat{\theta}_l(w, T(X, w, c_d), c_l) \\
\text{s.t.} \quad & T(X, w, c_d) = \arg\min_{X'} \widehat{\theta}_d(w, X', c_d)
\end{aligned}
$$

# The general problem

Defender (D) makes decision $\boldsymbol{\alpha} \in \mathbb{R}^n$. Attacker (A), after observing $\boldsymbol{\alpha}$, makes decision $\beta \in \mathbb{R}^m$

$$
\begin{aligned}
&\arg\max_{\alpha} \quad && u_D[\alpha, \beta^*(\alpha)] \\
&\text{s.t.} && \beta^*(\alpha) = \arg\max_{\beta} u_A(\alpha, \beta)
\end{aligned}
$$

- In AML, $\alpha$ and $\beta$ usually **high dimensional** and **continuous**.

# Gradient Methods

- Forget about analytical solutions!
- **Gradient methods** require computing $\mathrm{d}_\alpha u_D$ (and moving $\alpha$ in the direction of increasing gradient...)

$$
\begin{aligned}
\mathrm{d}_\alpha u_D &= \partial_\alpha u_D + \mathrm{d}_\alpha \beta^*(\alpha) \cdot \partial_\beta u_D \big|_{\beta^*(\alpha)} \\
&= \partial_\alpha u_D - \partial_\beta u_D \cdot \partial^2_{\alpha\beta} u_A \cdot \left[\partial^2_\beta u_A\right]^{-1} \big|_{\beta^*(\alpha)}
\end{aligned}
$$

- Inverting the Hessian has cubic complexity!
- We need a different strategy...

# Backward Solution

- Under **certain conditions** (Bottou, 1998), we can approximate our problem by

$$\begin{aligned} \arg\max_{\alpha} \quad & u_D\left[\alpha, \beta(\alpha, T)\right] \\ \text{s.t.} \quad & \partial_t \beta(\alpha, t) = \partial_\beta u_A[\alpha, \beta(\alpha, t)] \\ & \beta(\alpha, 0) = 0. \end{aligned}$$

- Where $T \gg 1$.
- $\lim_{t \to \infty} \beta(\alpha, t) = \beta^*(\alpha)$.
- Let's try to solve this problem instead.

# Backward Solution

- It can be proved that (Naveiro and Ríos, 2019)

$$\mathrm{d}_\alpha u_D[\alpha, \beta(\alpha, T)] = \partial_\alpha u_D[\alpha, \beta(\alpha, T)] - \int_0^T \lambda(t) \partial_\alpha \partial_\beta u_A[\alpha, \beta(\alpha, t)] \, \mathrm{d}t$$

- Provided that $\lambda$ satisfies the **adjoint equation**

$$\mathrm{d}_t \lambda(t) = -\lambda(t) \; \partial_\beta^2 u_A[\alpha, \beta(\alpha, t)]$$

- With initial conditions $\lambda(T) = -\partial_\beta u_D(\alpha, \beta)$.

# Backward Solution

**Algorithm 1** Approximate total derivative of defender utility function with respect to her decision using the backward solution

**procedure** APPROXIMATE DERIVATIVE USING BACKWARD METHOD$(\alpha, T)$

  $\beta_0(\alpha) = 0$

  **for** $t = 1, 2, \ldots, T$ **do**

    $\beta_t(\alpha) = \beta_{t-1}(\alpha) + \eta \partial_\beta u_A(\alpha, \beta)\Big|_{\beta_{t-1}}$

  **end for**

  $\lambda_T = -\partial_\beta u_D(\alpha, \beta)\Big|_{\beta_T}$

  $\mathrm{d}_\alpha u_D = \partial_\alpha u_D[\alpha, \beta_T(\alpha)]$

  **for** $t = T-1, T-2, \ldots, 0$ **do**

    $\mathrm{d}_\alpha u_D = \mathrm{d}_\alpha u_D - \eta \lambda_{t+1} \partial_\alpha \partial_\beta u_A(\alpha, \beta)\Big|_{\beta_{t+1}}$

    $\lambda_t = \lambda_{t+1} \left[ I + \eta \partial^2_\beta u_A(\alpha, \beta)\Big|_{\beta_{t+1}} \right]$

  **end for**

  **return** $\mathrm{d}_\alpha u_D$

**end procedure**

# Backward Solution - Complexity Analysis

## Time complexity

- If $\tau(n, m)$ is the time required to evaluate $u_D(\alpha, \beta)$ and $u_A(\alpha, \beta)$, computing their derivatives requires time $\mathcal{O}(\tau(n, m))$.
- First loop $\mathcal{O}(T\tau(n, m))$.
- Second loop needs computing Hessian Vector Products, by basic results of AD, they have same complexity as function evaluations!
- Thus, overall time complexity is $\mathcal{O}(T\tau(n, m))$.

## Space complexity

- We need to store $\beta_t(\alpha)$ for all $t$.
- $\sigma(n, m)$ is the space requirement for storing each $\beta_t(\alpha)$.
- Overall space complexity $\mathcal{O}(T\sigma(n, m))$.

# Forward Solution

- Under **certain conditions**, we can approximate our problem by

$$
\begin{aligned}
\arg\max_{\alpha} \quad & u_D\left[\alpha, \beta_T(\alpha)\right] \\
\text{s.t} \quad & \beta_t(\alpha) = \beta_{t-1}(\alpha) + \eta_t \partial_\beta u_A(\alpha, \beta)\Big|_{\beta_{t-1}} \qquad t = 1, \ldots, T \\
& \beta_0(\alpha) = 0.
\end{aligned}
$$

- Again, $T \gg 1$.
- $\lim_{t\to\infty} \beta_t(\alpha) = \beta^*(\alpha)$.

# Forward Solution

- Using the chain rule

$$\mathrm{d}_\alpha u_D[\alpha, \beta_T(\alpha)] = \partial_\alpha u_D[\alpha, \beta_T(\alpha)] + \partial_{\beta_T} u_D[\alpha, \beta_T(\alpha)]\, \mathrm{d}_\alpha \beta_T(\alpha)$$

- To obtain $\mathrm{d}_\alpha \beta_T(\alpha)$ we can sequentially compute

$$\mathrm{d}_\alpha \beta_t(\alpha) = \mathrm{d}_\alpha \beta_{t-1}(\alpha) + \eta_{t-1} \left[ \partial_\alpha \partial_\beta u_A(\alpha, \beta)\Big|_{\beta_{t-1}} + \partial_\beta^2 u_A(\alpha, \beta)\Big|_{\beta_{t-1}} \mathrm{d}_\alpha \beta_{t-1}(\alpha) \right]$$

- This induces a dynamical system in $\mathrm{d}_\alpha \beta_t(\alpha)$ that can be iterated in parallel to the dynamical system in $\beta_t(\alpha)$!

# Forward Solution

**Algorithm 2** Approximate total derivative of defender utility function with respect to her decision using the forward solution.

**procedure** APPROXIMATE DERIVATIVE USING FORWARD METHOD$(\alpha, T)$

$\beta_0(\alpha) = 0$

$\mathrm{d}_\alpha \beta_0(\alpha) = 0$

**for** $t = 1, 2, \ldots, T$ **do**

$$\beta_t(\alpha) = \beta_{t-1}(\alpha) + \eta \partial_\beta u_A(\alpha, \beta)\Big|_{\beta_{t-1}}$$

$$\mathrm{d}_\alpha \beta_t(\alpha) = \mathrm{d}_\alpha \beta_{t-1}(\alpha) + \eta_{t-1} \left[ \partial_\alpha \partial_\beta u_A(\alpha, \beta)\Big|_{\beta_{t-1}} + \partial_\beta^2 u_A(\alpha, \beta)\Big|_{\beta_{t-1}} \mathrm{d}_\alpha \beta_{t-1}(\alpha) \right]$$

**end for**

$\mathrm{d}_\alpha u_D = \partial_\alpha u_D[\alpha, \beta_T(\alpha)] + \partial_{\beta_T} u_D[\alpha, \beta_T(\alpha)] \, \mathrm{d}_\alpha \beta_T(\alpha)$

**return** $\mathrm{d}_\alpha u_D$

**end procedure**

# Forward Solution - Complexity Analysis

## Time complexity

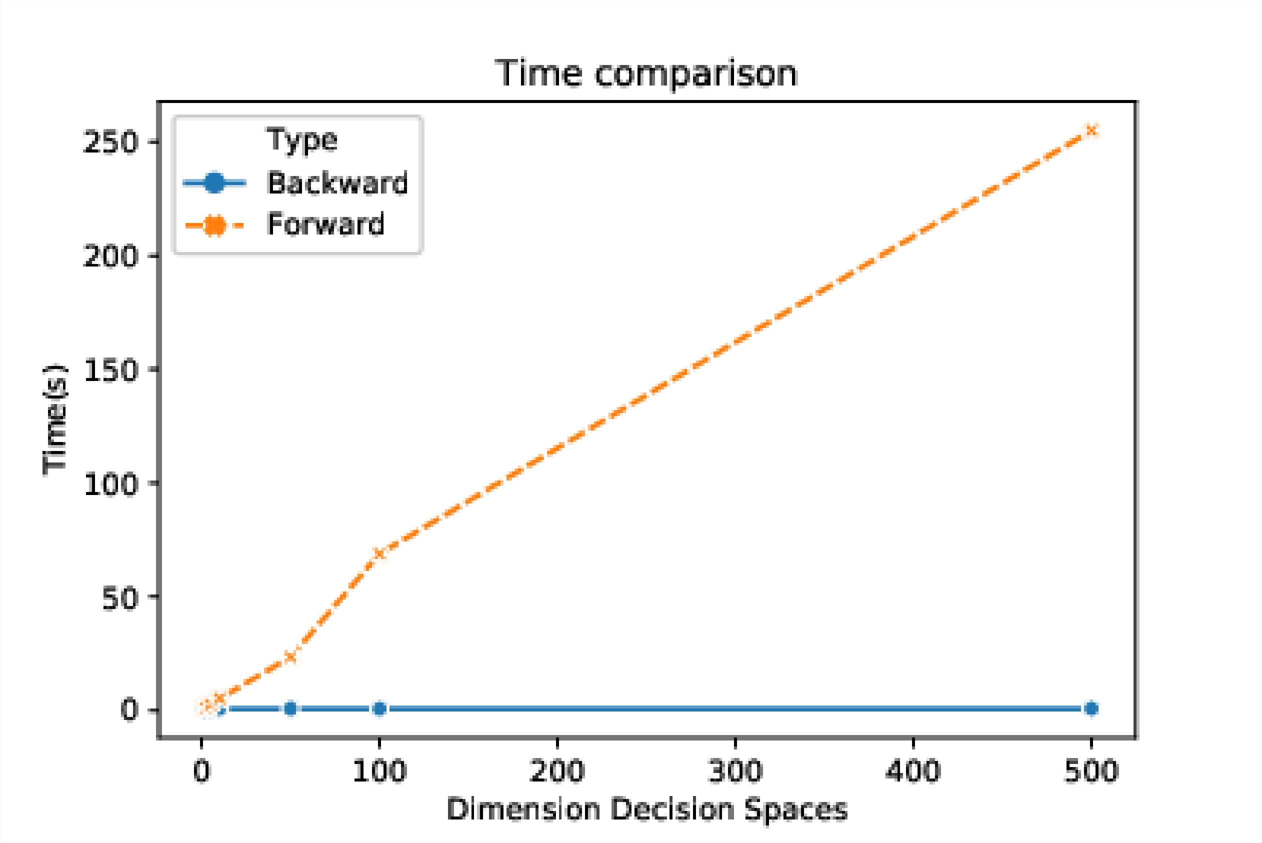
- Computing $\partial_\beta^2 u_A(\alpha, \beta)$ requires time $\mathcal{O}(m\tau(m,n))$ as it requires computing $m$ Hessian vector products.
- Computing $\partial_\alpha \partial_\beta u_A(\alpha, \beta)$ requires computing $n$ Hessian vector products and thus time $\mathcal{O}(n\tau(m,n))$.
- If we compute the derivative in the other way, first we derive with respect to $\beta$ and then with respect to $\alpha$, the time complexity is $\mathcal{O}(m\tau(m,n))$.
- Thus, computing $\partial_\alpha \partial_\beta u_A(\alpha, \beta)$ requires $\mathcal{O}(\min(n,m)\tau(m,n))$.
- Overall, $\mathcal{O}(\max[\min(n,m), m]T\tau(m,n)) = \mathcal{O}(mT\tau(m,n))$.

## Space complexity

- The values $\beta_t(\alpha)$ are overwritten at each iteration.
- Overall space complexity is $\mathcal{O}(\sigma(m,n))$.

# Conceptual Example

- Attacker's utility is $u_A(\alpha, \beta) = -\sum_{i=1}^{n} 3(\beta_i - \alpha_i)^2$ and the defender's one is $u_D(\alpha, \beta) = -\sum_{i=1}^{n}(7\alpha_i + \beta_i^2)$.
- $\mathcal{O}(T\tau(m,n))$ vs $\mathcal{O}(mT\tau(m,n))$.



# Application - Adversarial Regression

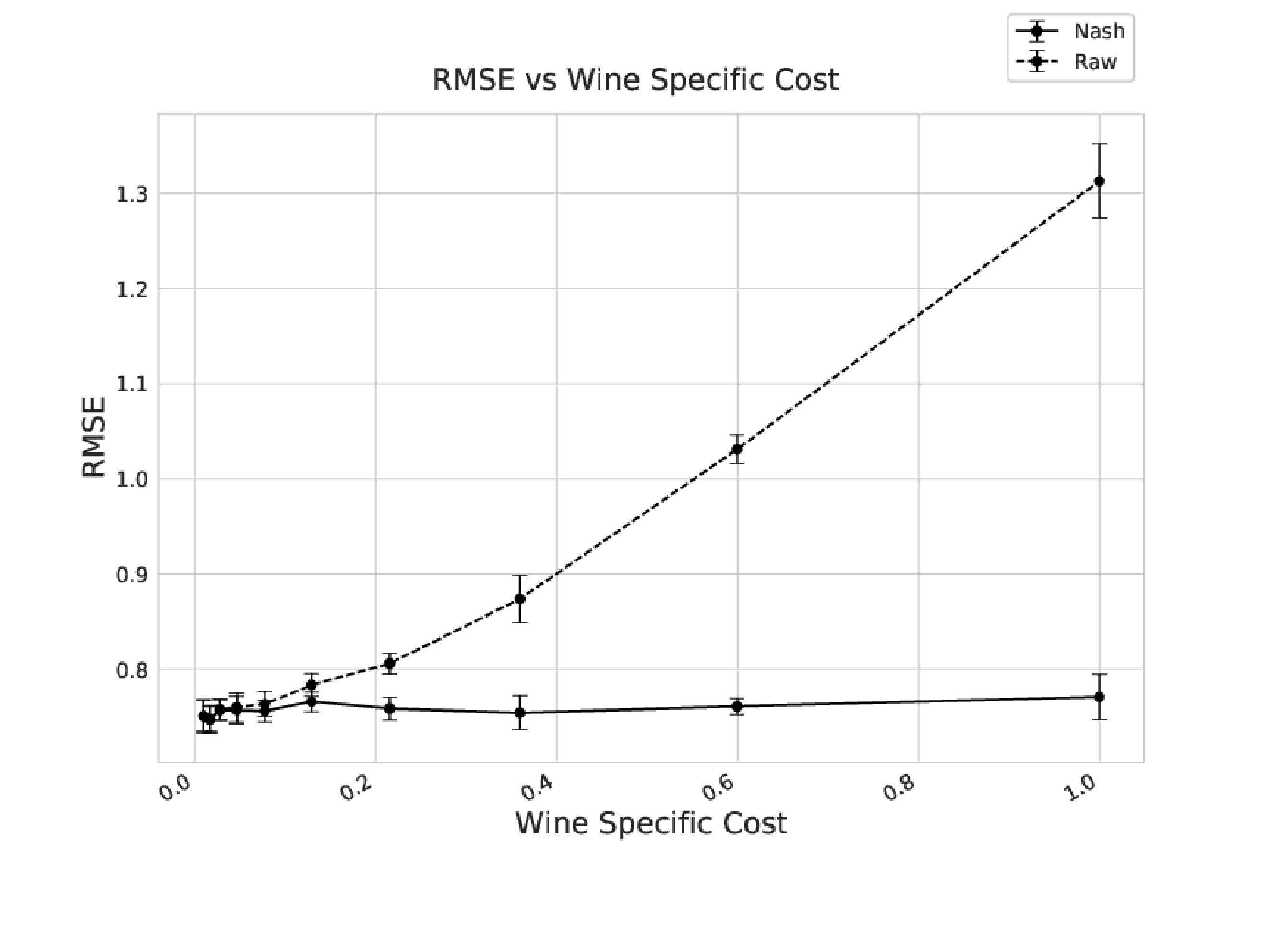
- We compare ridge regression versus *adversarial robust regression* in the wine problem.
- For ridge regression, we compute the weights in the usual way, and test them in data attacked using those weights.
- For *adversarial robust regression* we compute the weights solving

$$
\begin{aligned}
\arg\min_{w} \quad & \widehat{\theta}_l(w, T(X, w, c_d), c_l) \\
\text{s.t.} \quad & T(X, w, c_d) = \arg\min_{X'} \widehat{\theta}_d(w, X', c_d)
\end{aligned}
$$

and test them in data attacked using those weights.

- Note the dimension of the attacker's decision space is huge! He needs to modify $\boldsymbol{k = 3263}$ data points each with $\boldsymbol{n = 11}$ components!

# Adversarial Regression



# Conclusions and future work

- New algorithmic method able to solve **huge Stackelberg Games** (dimension of decision sets of the order of $10^4$).
- Could be implemented in any **Automatic Differentiation** library (Pytorch, tensorflow...).
- Novel derivation of the backward solution formulating the Stackelberg game as a PDE-constrained optimization problem.
- Application to games with uncertain outcomes.
- Application to Bayesian Stackelberg Games and ARA.
- Several attackers?

# Thank you!!

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www.github.com/roinaveiro/GM_SG

www.roinaveiro.github.io